

MINIMUM DISTANCE FUNCTIONS OF COMPLETE INTERSECTIONS

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ABSTRACT. We study the *minimum distance function* of a complete intersection graded ideal in a polynomial ring with coefficients in a field. For graded ideals of dimension one, whose initial ideal is a complete intersection, we use the footprint function to give a sharp lower bound for the minimum distance function. Then we show some applications to coding theory.

1. INTRODUCTION

Let $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and let $I \neq (0)$ be a graded ideal of S . The *degree* or *multiplicity* of S/I is denoted by $\deg(S/I)$. Fix a graded monomial order \prec on S and let $\text{in}_{\prec}(I)$ be the initial ideal of I .

The *footprint* of S/I or *Gröbner escalier* of I , denoted $\Delta_{\prec}(I)$, is the set of all monomials of S not in the ideal $\text{in}_{\prec}(I)$ [28, p. 13, p. 133]. This notion occurs in other branches of mathematics under different names; see [20, p. 6] for a list of alternative names.

Given an integer $d \geq 1$, let $\mathcal{M}_{\prec,d}$ be the set of all zero-divisors of $S/\text{in}_{\prec}(I)$ of degree d that are in $\Delta_{\prec}(I)$, and let $\mathcal{F}_{\prec,d}$ be the set of all zero-divisors of S/I that are not zero and are a K -linear combination of monomials in $\Delta_{\prec}(I)$ of degree d .

The *footprint function* of I , denoted fp_I , is the function $\text{fp}_I: \mathbb{N}_+ \rightarrow \mathbb{Z}$ given by

$$\text{fp}_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(\text{in}_{\prec}(I), t^a)) \mid t^a \in \mathcal{M}_{\prec,d}\} & \text{if } \mathcal{M}_{\prec,d} \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{M}_{\prec,d} = \emptyset, \end{cases}$$

and the *minimum distance function* of I , denoted δ_I , is the function $\delta_I: \mathbb{N}_+ \rightarrow \mathbb{Z}$ given by

$$\delta_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(I, f)) \mid f \in \mathcal{F}_{\prec,d}\} & \text{if } \mathcal{F}_{\prec,d} \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{F}_{\prec,d} = \emptyset. \end{cases}$$

These two functions were introduced and studied in [24]. Notice that δ_I is independent of the monomial order \prec (see Lemma 3.9). To compute δ_I is a difficult problem but to compute fp_I is much easier.

We come to the main result of this paper which gives an explicit lower bound for δ_I and a formula for fp_I for a family of complete intersection graded ideals:

Theorem 3.14 *If the initial ideal $\text{in}_{\prec}(I)$ of I is a complete intersection of height $s-1$ generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then*

$$\delta_I(d) \geq \text{fp}_I(d) = \begin{cases} (d_{k+2} - \ell)d_{k+3} \cdots d_s & \text{if } d \leq \sum_{i=2}^s (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum_{i=2}^s (d_i - 1), \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

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An important case of this theorem, from the viewpoint of applications, is when I is the vanishing ideal of a finite set of projective points over a finite field (see the discussion below about the connection of fp_I and δ_I with coding theory). If I is a complete intersection monomial ideal of dimension 1, then $\delta_I(d) = \text{fp}_I(d)$ for $d \geq 1$ (see Proposition 3.11), but this case is only of theoretical interest because, by Proposition 2.16, a monomial ideal is a vanishing ideal only in particular cases.

Let $I \subset S$ be a graded ideal such that $L = \text{in}_{\prec}(I)$ is a complete intersection of dimension 1. We give a formula for the degree of $S/(L, t^a)$ when t^a is in $\mathcal{M}_{\prec, d}$, that is, t^a is not in L and is a zero-divisor of S/L . By an easy classification of the complete intersection property of L (see Lemma 3.1) there are basically two cases to consider. One of them is Lemma 3.4, and the other is the following:

Lemma 3.3 *If $L = \text{in}_{\prec}(I)$ is generated by $t_2^{d_2}, \dots, t_s^{d_s}$ and $t^a = t_1^{a_1} \cdots t_s^{a_s}$ is in $\mathcal{M}_{\prec, d}$, then*

$$\deg(S/(L, t^a)) = d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s).$$

To show our main result we use the formula for the degree of the ring $S/(L, t^a)$, and then use Proposition 3.13 to bound the degrees uniformly. The proof of the main result takes place in an abstract algebraic setting with no reference to vanishing ideals or finite fields.

The formulas for the degree are useful in the following setting. If $I = I(\mathbb{X})$ is the vanishing ideal of a finite set \mathbb{X} of projective points, and $\text{in}_{\prec}(I)$ is generated by $t_2^{d_2}, \dots, t_s^{d_s}$, then Lemma 3.3 can be used to give upper bounds for the number of zeros in \mathbb{X} of homogeneous polynomials of S . In fact, if $f \in \mathcal{F}_{\prec, d}$ and $\text{in}_{\prec}(f) = t_1^{a_1} \cdots t_s^{a_s}$, then $\text{in}_{\prec}(f)$ is in $\mathcal{M}_{\prec, d}$, and by Corollary 3.8 one has:

$$|V_{\mathbb{X}}(f)| \leq d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s),$$

where $V_{\mathbb{X}}(f)$ is the set of zeros or variety of f in \mathbb{X} . This upper bound depends on the exponent of the leading term of f . A more complex upper bound is obtained when the initial ideal of $I(\mathbb{X})$ is as in Lemma 3.1(ii). In this case one uses the formula for the degree given in Lemma 3.4.

The interest in studying fp_I and δ_I comes from algebraic coding theory. Indeed, if $I = I(\mathbb{X})$ is the vanishing ideal of a finite subset \mathbb{X} of a projective space \mathbb{P}^{s-1} over a finite field $K = \mathbb{F}_q$, then the minimum distance $\delta_{\mathbb{X}}(d)$ of the corresponding projective Reed-Muller-type code is equal to $\delta_{I(\mathbb{X})}(d)$, and $\text{fp}_{I(\mathbb{X})}(d)$ is a lower bound for $\delta_{\mathbb{X}}(d)$ for $d \geq 1$ (see Theorem 2.13 and Lemma 3.10). Therefore, one has the formula:

$$(2.2) \quad \delta_{I(\mathbb{X})}(d) = \deg(S/I(\mathbb{X})) - \max\{|V_{\mathbb{X}}(f)| : f \neq 0; f \in S_d\},$$

where $f \neq 0$ means that f is not the zero function on \mathbb{X} . Our abstract study of the minimum distance and footprint functions provides fresh techniques to study $\delta_{\mathbb{X}}(d)$.

It is well-known that the degree of $S/I(\mathbb{X})$ is equal to $|\mathbb{X}|$ [18, Lecture 13]. Hence, using Eq. (2.2) and our main result, we get the following uniform upper bound for the number of zeros of all polynomials $f \in S_d$ that do not vanish at all points of \mathbb{X} .

Corollary 4.2 *If the initial ideal $\text{in}_{\prec}(I(\mathbb{X}))$ is a complete intersection generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then*

$$(4.1) \quad |V_{\mathbb{X}}(f)| \leq |\mathbb{X}| - (d_{k+2} - \ell) d_{k+3} \cdots d_s,$$

for any $f \in S_d$ that does not vanish at all point of \mathbb{X} , where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

This result gives a tool for finding good uniform upper bounds for the number of zeros in \mathbb{X} of polynomials over finite fields. This is a problem of fundamental interest in algebraic coding

theory [31] and algebraic geometry [30]. We leave as an open question whether this uniform bound is optimal, that is, whether the equality is attained for some polynomial f .

Tohăneanu and Van Tuyl conjectured [33, Conjecture 4.9] that if the vanishing ideal $I(\mathbb{X})$ is a complete intersection generated by polynomials of degrees d_2, \dots, d_s and $d_i \leq d_{i+1}$ for all i , then $\delta_{\mathbb{X}}(1) \geq (d_2 - 1)d_3 \dots d_s$. By Corollary 4.2 this conjecture is true if $\text{in}_{\prec}(I(\mathbb{X}))$ is a complete intersection. We leave as another open question whether Corollary 4.2 is true if we only assume that $I(\mathbb{X})$ is a complete intersection (cf. Proposition 3.12).

To illustrate the use of Corollary 4.2 in a concrete situation, consider the lexicographical order on S with $t_1 \prec \dots \prec t_s$ and a *projective torus* over a finite field \mathbb{F}_q with $q \neq 2$:

$$\mathbb{T} = \{(x_1, \dots, x_s) \in \mathbb{P}^{s-1} \mid x_i \in \mathbb{F}_q^* \text{ for } i = 1, \dots, s\},$$

where $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. As $I(\mathbb{T})$ is generated by the Gröbner basis $\{t_i^{q-1} - t_1^{q-1}\}_{i=2}^s$, its initial ideal is a complete intersection generated by $t_2^{q-1}, \dots, t_s^{q-1}$. Therefore, noticing that $\deg(S/I(\mathbb{T}))$ is equal to $(q-1)^{s-1}$ and setting $d_i = q-1$ for $i = 2, \dots, s$ in Eq. (4.1), we obtain that any homogeneous polynomial f of degree d , not vanishing at all points of \mathbb{T} , has at most

$$(q-1)^{s-1} - (q-1)^{s-(k+2)}(q-1-\ell)$$

zeros in \mathbb{T} if $d \leq (q-2)(s-1) - 1$, and k and ℓ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-2$ and $d = k(q-2) + \ell$. This uniform bound was given in [29, Theorem 3.5] and it is seen that this bound is in fact optimal by constructing an appropriate polynomial f .

If $\text{fp}_I(d) = \delta_I(d)$ for $d \geq 1$, we say that I is a *Geil–Carvalho ideal*. For vanishing ideals over finite fields, this notion is essentially another way of saying that the bound of Eq. (4.1) is optimal. The first interesting family of ideals where equality holds is due to Geil [12, Theorem 2]. His result essentially shows that $\text{fp}_I(d) = \delta_I(d)$ for $d \geq 1$ when \prec is a graded lexicographical order and I is the homogenization of the vanishing ideal of the affine space \mathbb{A}^{s-1} over a finite field $K = \mathbb{F}_q$. Recently, Carvalho [4, Proposition 2.3] extended this result by replacing \mathbb{A}^{s-1} by a cartesian product of subsets of \mathbb{F}_q . In this case the underlying Reed-Muller-type code is called an affine cartesian code and an explicit formula for the minimum distance was first given in [13, 21]. In a very recent paper, Bishnoi, Clark, Potukuchi, and Schmitt give another proof of this formula [3, Theorem 5.2] using a result of Alon and Füredi [1, Theorem 5].

As the two most relevant applications of our main result to algebraic coding theory, we recover the formula for the minimum distance of an affine cartesian code given in [21, Theorem 3.8] and [13, Proposition 5], and the fact that the homogenization of the corresponding vanishing ideal is a Geil–Carvalho ideal [4] (see Corollary 4.4).

Then we present an extension of a result of Alon and Füredi [1, Theorem 1]—in terms of the regularity of a vanishing ideal—about coverings of the cube $\{0, 1\}^n$ by affine hyperplanes, that can be applied to any finite subset of a projective space whose vanishing ideal has a complete intersection initial ideal (see Corollary 4.5 and Example 4.6).

Finally, using *Macaulay2* [16], we exemplify how some of our results can be used in practice, and show that the vanishing ideal of \mathbb{P}^2 over \mathbb{F}_2 is not Geil–Carvalho by computing all possible initial ideals (see Example 4.8).

In Section 2 we introduce projective Reed-Muller-type codes and present some of the results and terminology that will be needed in the paper. For all unexplained terminology and additional information, we refer to [38] (for deeper advances on the knowledge of the degree), [7] (for the theory of Gröbner bases), [2, 10, 32] (for commutative algebra and Hilbert functions), and [23, 34] (for the theory of error-correcting codes and linear codes).

2. PRELIMINARIES

In this section, we present some of the results that will be needed throughout the paper and introduce some more notation. All results of this section are well-known.

Let $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ be a graded polynomial ring over a field K with the standard grading and let $(0) \neq I \subset S$ be a graded ideal. The *Hilbert function* of S/I is:

$$H_I(d) := \dim_K(S_d/I_d), \quad d = 0, 1, 2, \dots$$

where $I_d = I \cap S_d$. By the dimension of I we mean the Krull dimension of S/I .

Theorem 2.1. (Hilbert [32, p. 58]) *Let $I \subset S$ be a graded ideal of dimension k . Then there is a polynomial $h_I(x) \in \mathbb{Q}[x]$ of degree $k - 1$ such that $h_I(d) = H_I(d)$ for $d \gg 0$.*

The *degree* or *multiplicity* of S/I is the positive integer

$$\deg(S/I) := \begin{cases} (k-1)! \lim_{d \rightarrow \infty} H_I(d)/d^{k-1} & \text{if } k \geq 1, \\ \dim_K(S/I) & \text{if } k = 0. \end{cases}$$

If $f \in S$, the *quotient ideal* of I with respect to f is given by $(I : f) = \{h \in S \mid hf \in I\}$. The element f is called a *zero-divisor* of S/I —as an S -module—if there is $\bar{0} \neq \bar{a} \in S/I$ such that $f\bar{a} = \bar{0}$, and f is called *regular* on S/I otherwise. Notice that f is a zero-divisor of S/I if and only if $(I : f) \neq I$. An associated prime of I is a prime ideal \mathfrak{p} of S of the form $\mathfrak{p} = (I : f)$ for some f in S .

Definition 2.2. The *regularity of the Hilbert function* of S/I , or simply the *regularity* of S/I , denoted $\text{reg}(S/I)$, is the least integer $r \geq 0$ such that $H_I(d)$ is equal to $h_I(d)$ for $d \geq r$.

If S/I is a Cohen–Macaulay ring of dimension 1, then $\text{reg}(S/I)$ is precisely the Castelnuovo–Mumford regularity of S/I in the sense of [10, p. 509]. This follows readily from a well-known formula that relates the a -invariant, the Castelnuovo–Mumford regularity, and the depth of the ring S/I (see for instance [36, Corollary B.4.1]).

Definition 2.3. An ideal $I \subset S$ is called a *complete intersection* if there exist g_1, \dots, g_r in S such that $I = (g_1, \dots, g_r)$, where r is the height of I .

Remark 2.4. Let $I \subset S$ be a graded ideal. (a) Any two minimal sets of homogeneous generators of I have the same cardinality. (b) I is a complete intersection if and only if I is minimally generated by a homogeneous regular sequence with $\text{ht}(I)$ elements, where $\text{ht}(I)$ denotes the height of I (see [19, Chapter 3]). (c) A monomial ideal I is a complete intersection if and only if I is minimally generated by a regular sequence of monomials with $\text{ht}(I)$ elements.

If I is a graded ideal of S , the *Hilbert series* of S/I , denoted $F_I(x)$, is given by

$$F_I(x) = \sum_{d=0}^{\infty} H_I(d)x^d, \quad \text{where } x \text{ is a variable.}$$

Theorem 2.5. (Hilbert–Serre [32, p. 58], [35, p. 297]) *Let $I \subset S$ be a graded ideal of dimension k . Then there is a unique polynomial $h(x) \in \mathbb{Z}[x]$ such that*

$$F_I(x) = \frac{h(x)}{(1-x)^k} \quad \text{and} \quad h(1) > 0.$$

Remark 2.6. The leading coefficient of the Hilbert polynomial $h_I(x)$ is equal to $h(1)/(k-1)!$. Thus $h(1)$ is equal to $\deg(S/I)$.

Lemma 2.7. [32, Corollary 3.3] *If $I \subset S$ is an ideal generated by homogeneous polynomials f_1, \dots, f_r , with $r = \text{ht}(I)$ and $\delta_i = \deg(f_i)$, then*

$$F_I(x) = \frac{\prod_{i=1}^r (1 - x^{\delta_i})}{(1 - x)^s} \quad \text{and} \quad \deg(S/I) = \delta_1 \cdots \delta_r.$$

The footprint of an ideal. Let \prec be a monomial order on S and let $(0) \neq I \subset S$ be an ideal. If f is a non-zero polynomial in S , one can write

$$f = \lambda_1 t^{\alpha_1} + \cdots + \lambda_r t^{\alpha_r},$$

with $\lambda_i \in K^* = K \setminus \{0\}$ for all i and $t^{\alpha_1} \succ \cdots \succ t^{\alpha_r}$. The *leading monomial* t^{α_1} of f is denoted by $\text{in}_{\prec}(f)$. The *initial ideal* of I , denoted by $\text{in}_{\prec}(I)$, is the monomial ideal of S given by

$$\text{in}_{\prec}(I) := (\{\text{in}_{\prec}(f) \mid f \in I\}).$$

A monomial t^a is called a *standard monomial* of S/I , with respect to \prec , if t^a is not the leading monomial of any polynomial in I , that is, t^a is not in the ideal $\text{in}_{\prec}(I)$. A polynomial f is called *standard* if $f \neq 0$ and f is a K -linear combination of standard monomials.

The set of standard monomials, denoted $\Delta_{\prec}(I)$, is called the *footprint* of S/I or *Gröbner escalier* of I . The image of $\Delta_{\prec}(I)$, under the canonical map $S \mapsto S/I$, $x \mapsto \overline{x}$, is a basis of S/I as a K -vector space. This is a classical result of Macaulay (for a modern approach see [7, Chapter 5, Section 3, Proposition 4]). In particular, if I is graded, then $H_I(d)$ is the number of standard monomials of degree d .

A subset $\mathcal{G} = \{g_1, \dots, g_r\}$ of I is called a *Gröbner basis* of I if

$$\text{in}_{\prec}(I) = (\text{in}_{\prec}(g_1), \dots, \text{in}_{\prec}(g_r)).$$

Lemma 2.8. *Let \prec be a monomial order, let $I \subset S$ be an ideal, and let f be a polynomial of S of positive degree. If $\text{in}_{\prec}(f)$ is regular on $S/\text{in}_{\prec}(I)$, then f is regular on S/I .*

Proof. Let g be a polynomial of S such that $gf \in I$. It suffices to show that $g \in I$. Pick a Gröbner basis g_1, \dots, g_r of I . Then, by the division algorithm [7, Theorem 3, p. 63], we can write $g = \sum_{i=1}^r a_i g_i + h$, where $h = 0$ or h is a standard polynomial of S/I . We need only show that $h = 0$. If $h \neq 0$, then hf is in I and $\text{in}_{\prec}(h)\text{in}_{\prec}(f)$ is in $\text{in}_{\prec}(I)$. Therefore $\text{in}_{\prec}(h)$ is in $\text{in}_{\prec}(I)$, a contradiction. \square

Remark 2.9. Given an integer $d \geq 1$, there is a map $\text{in}_{\prec}: \mathcal{F}_{\prec,d} \rightarrow \mathcal{M}_{\prec,d}$ given by $f \mapsto \text{in}_{\prec}(f)$. This follows from Lemma 2.8. If I is a monomial ideal, then $\mathcal{M}_{\prec,d} \subset \mathcal{F}_{\prec,d}$.

Projective Reed-Muller-type codes. Let $K = \mathbb{F}_q$ be a finite field with q elements, let \mathbb{P}^{s-1} be a projective space over K , and let \mathbb{X} be a subset of \mathbb{P}^{s-1} . As usual, points of \mathbb{P}^{s-1} are denoted by $[\alpha]$, where $0 \neq \alpha \in K^s$. In this paragraph all results are valid if we assume that K is any field and \mathbb{X} is a finite subset of \mathbb{P}^{s-1} , instead of assuming that K is finite. However, the interesting case for coding theory is when K is finite.

The *vanishing ideal* of \mathbb{X} , denoted $I(\mathbb{X})$, is the ideal of S generated by the homogeneous polynomials that vanish at all points of \mathbb{X} . In this case the Hilbert function of $S/I(\mathbb{X})$ is denoted by $H_{\mathbb{X}}(d)$. Let P_1, \dots, P_m be a set of representatives for the points of \mathbb{X} with $m = |\mathbb{X}|$. Fix a degree $d \geq 1$. For each i there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. Indeed suppose $P_i = [(a_1, \dots, a_s)]$, there is at least one k in $\{1, \dots, s\}$ such that $a_k \neq 0$. Setting $f_i(t_1, \dots, t_s) = t_k^d$ one has that $f_i \in S_d$ and $f_i(P_i) \neq 0$. There is a K -linear map:

$$(2.1) \quad \text{ev}_d: S_d = K[t_1, \dots, t_s]_d \rightarrow K^{|\mathbb{X}|}, \quad f \mapsto \left(\frac{f(P_1)}{f_1(P_1)}, \dots, \frac{f(P_m)}{f_m(P_m)} \right).$$

The map ev_d is called an *evaluation map*. The image of S_d under ev_d , denoted by $C_{\mathbb{X}}(d)$, is called a *projective Reed-Muller-type code* of degree d over \mathbb{X} [9]. It is also called an *evaluation code* associated to \mathbb{X} [14]. This type of codes have been studied using commutative algebra methods and especially Hilbert functions, see [8, 15, 26, 31] and the references therein.

Definition 2.10. A *linear code* is a linear subspace of K^m for some m . The *basic parameters* of the linear code $C_{\mathbb{X}}(d)$ are its *length* $|\mathbb{X}|$, *dimension* $\dim_K C_{\mathbb{X}}(d)$, and *minimum distance*

$$\delta_{\mathbb{X}}(d) := \min\{\|v\| : 0 \neq v \in C_{\mathbb{X}}(d)\},$$

where $\|v\|$ is the number of non-zero entries of v .

Lemma 2.11. [24, Lemma 2.13] (a) *The map ev_d is well-defined, i.e., it is independent of the set of representatives that we choose for the points of \mathbb{X} .* (b) *The basic parameters of the Reed-Muller-type code $C_{\mathbb{X}}(d)$ are independent of f_1, \dots, f_m .*

The following summarizes the well-known relation between projective Reed-Muller-type codes and the theory of Hilbert functions. Notice that items (i) and (iv) follow directly from Eq. (2.1) and item (iii), respectively.

Proposition 2.12. *The following hold.*

- (i) $H_{\mathbb{X}}(d) = \dim_K C_{\mathbb{X}}(d)$ for $d \geq 1$.
- (ii) [18, Lecture 13] $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$.
- (iii) (*Singleton bound*) $1 \leq \delta_{\mathbb{X}}(d) \leq |\mathbb{X}| - H_{\mathbb{X}}(d) + 1$ for $d \geq 1$.
- (iv) $\delta_{\mathbb{X}}(d) = 1$ for $d \geq \text{reg}(S/I(\mathbb{X}))$.

The next result gives an algebraic formulation of the minimum distance of a projective Reed-Muller-type code in terms of the degree and the structure of the underlying vanishing ideal.

Theorem 2.13. [24, Theorem 4.7] *If $|\mathbb{X}| \geq 2$, then $\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \geq 1$ for $d \geq 1$.*

This result gives an algorithm, that can be implemented in *CoCoA* [6], *Macaulay2* [16], or *Singular* [17], to compute $\delta_{\mathbb{X}}(d)$ for small values of q and s , where q is the cardinality of \mathbb{F}_q and s is the number of variables of S (see the procedure of Example 4.6). Using SAGE [27] one can also compute $\delta_{\mathbb{X}}(d)$ by finding a generator matrix of $C_{\mathbb{X}}(d)$.

As a direct consequence of Theorem 2.13 one has:

$$(2.2) \quad \delta_{I(\mathbb{X})}(d) = \deg(S/I(\mathbb{X})) - \max\{|V_{\mathbb{X}}(f)| : f \neq 0; f \in S_d\},$$

where $V_{\mathbb{X}}(f)$ is the zero set of f in \mathbb{X} and $f \neq 0$ means that f does not vanish at all points of \mathbb{X} .

The next lemma follows using the division algorithm [7] (cf. [11, Problem 1-17]).

Lemma 2.14. *Let \mathbb{X} be a finite subset of \mathbb{P}^{s-1} , let $[\alpha]$ be a point in \mathbb{X} , with $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\alpha_k \neq 0$ for some k , and let $I_{[\alpha]}$ be the vanishing ideal of $[\alpha]$. Then $I_{[\alpha]}$ is a prime ideal,*

$$I_{[\alpha]} = (\{\alpha_k t_i - \alpha_i t_k \mid k \neq i \in \{1, \dots, s\}\}), \quad \deg(S/I_{[\alpha]}) = 1,$$

$\text{ht}(I_{[\alpha]}) = s - 1$, and $I(\mathbb{X}) = \bigcap_{[\beta] \in \mathbb{X}} I_{[\beta]}$ is the primary decomposition of $I(\mathbb{X})$.

Remark 2.15. If \mathbb{X} is a finite set of projective points, then $S/I(\mathbb{X})$ is a Cohen–Macaulay reduced graded ring of dimension 1. This is very well-known, and it follows directly from Lemma 2.14. In particular, the regularity of the Hilbert function of $S/I(\mathbb{X})$ is the Castelnuovo–Mumford regularity of $S/I(\mathbb{X})$.

The next result classifies monomial vanishing ideals of finite sets in a projective space.

Proposition 2.16. *Let \mathbb{X} be a finite subset of \mathbb{P}^{s-1} . The following are equivalent:*

- (a) $I(\mathbb{X})$ is a monomial ideal.
- (b) $I(\mathbb{X}) = \cap_{i=1}^m \mathfrak{p}_i$, where the \mathfrak{p}_i 's are generated by $s-1$ variables.
- (c) $\mathbb{X} \subset \{[e_1], \dots, [e_s]\}$, where e_i is the i -th unit vector.

Proof. (a) \Rightarrow (b): By Remark 2.15, $I(\mathbb{X})$ is a radical Cohen–Macaulay graded ideal of dimension 1. Hence, $I(\mathbb{X})$ is an unmixed square-free monomial ideal of height $s-1$. Therefore, $I(\mathbb{X})$ is equal to $\cap_{i=1}^m \mathfrak{p}_i$, where the \mathfrak{p}_i 's are face ideals (i.e., ideals generated by variables) of height $s-1$.

(b) \Rightarrow (c): Let $\overline{\mathbb{X}}$ be the Zariski closure of \mathbb{X} . As \mathbb{X} is finite, one has $\mathbb{X} = \overline{\mathbb{X}} = V(I(\mathbb{X})) = \cup_{i=1}^m V(\mathfrak{p}_i)$. Thus it suffices to notice that $V(\mathfrak{p}_i) = \{[e_{i_j}]\}$ for some i_j .

(c) \Rightarrow (a): This follows from Lemma 2.14. \square

3. COMPLETE INTERSECTIONS

Let $S = K[t_1, \dots, t_s] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and $s \geq 2$. In what follows by a monomial order \prec we mean a graded monomial order in the sense that \prec is defined first by total degree [7].

Lemma 3.1. *Let $L \subset S$ be an ideal generated by monomials. If $\dim(S/L) = 1$, then L is a complete intersection if and only if, up to permutation of the variables t_1, \dots, t_s , we can write*

- (i) $L = (t_2^{d_2}, \dots, t_s^{d_s})$ with $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, or
- (ii) $L = (t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p} t_{p+1}^{c_{p+1}}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s})$ for some $p \geq 1$ such that $1 \leq c_p \leq c_{p+1}$ and $1 \leq d_i \leq d_{i+1}$ for $2 \leq i \leq s-1$, where $d_{p+1} = c_p + c_{p+1}$.

Proof. \Rightarrow) Let $t^{\alpha_1}, \dots, t^{\alpha_{s-1}}$ be the minimal set of generators of L consisting of monomials. These monomials form a regular sequence (see Remark 2.4). Hence t^{α_i} and t^{α_j} have no common variables for $i \neq j$. Then, either all variables occur in $t^{\alpha_1}, \dots, t^{\alpha_{s-1}}$ and we are in case (ii), up to permutation of the variables t_1, \dots, t_s , or there is one variable that is not in any of the t^{α_i} 's and we are in case (i), up to permutation of the variables t_1, \dots, t_s .

\Leftarrow) In both cases L is an ideal of height $s-1$ generated by $s-1$ elements, that is, L is a complete intersection. \square

Proposition 3.2. [37, Propositions 3.1.33 and 5.1.11] *Let $A = R_1/I_1$, $B = R_2/I_2$ be two standard graded algebras over a field K , where $R_1 = K[\mathbf{x}]$, $R_2 = K[\mathbf{y}]$ are polynomial rings in disjoint sets of variables and I_i is an ideal of R_i . If $R = K[\mathbf{x}, \mathbf{y}]$ and $I = I_1R + I_2R$, then*

$$(R_1/I_1) \otimes_K (R_2/I_2) \simeq R/I \quad \text{and} \quad F(A \otimes_K B, x) = F(A, x)F(B, x),$$

where $F(A, x)$ and $F(B, x)$ are the Hilbert series of A and B , respectively.

Lemma 3.3. *Let L be the ideal of S generated by $t_2^{d_2}, \dots, t_s^{d_s}$. If $t^a = t_1^{a_1} t_r^{a_r} \dots t_s^{a_s}$, $r \geq 2$, $a_r \geq 1$, and $a_i \leq d_i - 1$ for $i \geq r$, then*

$$\deg(S/(L, t^a)) = \deg(S/(L, t_r^{a_r} \dots t_s^{a_s})) = d_2 \dots d_s - d_2 \dots d_{r-1} (d_r - a_r) \dots (d_s - a_s),$$

where $a_i = 0$ if $2 \leq i < r$.

Proof. In what follows we will use the fact that Hilbert functions and Hilbert series are additive on short exact sequences [11, Chapter 2, Proposition 7]. If $a_1 \geq 1$, then taking Hilbert functions in the exact sequence

$$0 \longrightarrow S/(L, t_r^{a_r} \dots t_s^{a_s})[-a_1] \xrightarrow{t_1^{a_1}} S/(L, t^a) \longrightarrow S/(L, t_1^{a_1}) \longrightarrow 0,$$

and noticing that $\dim(S/(L, t_1^{a_1})) = 0$, the first equality follows. Thus we may assume that t^a has the form $t^a = t_r^{a_r} \cdots t_s^{a_s}$ and $a_i = 0$ for $i < r$.

We proceed by induction on $s \geq 2$. Assume $s = 2$. Then $r = 2$, $t^a = t_2^{a_2}$, $(L, t^a) = (t_2^{a_2})$, and the degree of $S/(L, t^a)$ is a_2 , as required. Assume $s \geq 3$. If $a_i = 0$ for $i > r$, then $(L, t^a) = (L, t_r^{a_r})$ is a complete intersection and the required formula follows from Lemma 2.7. Thus we may assume that $a_i \geq 1$ for some $i > r$. There is an exact sequence

$$(3.1) \quad 0 \longrightarrow S/(t_2^{d_2}, \dots, t_{r-1}^{d_{r-1}}, t_r^{d_r - a_r}, t_{r+1}^{d_{r+1}}, \dots, t_s^{d_s}, t_{r+1}^{a_{r+1}} \cdots t_s^{a_s})[-a_r] \xrightarrow{t_r^{a_r}} S/(L, t^a) \longrightarrow S/(t_2^{d_2}, \dots, t_{r-1}^{d_{r-1}}, t_r^{a_r}, t_{r+1}^{d_{r+1}}, \dots, t_s^{d_s}) \longrightarrow 0.$$

Notice that the ring on the right is a complete intersection and the ring on the left is isomorphic to the tensor product

$$(3.2) \quad K[t_2, \dots, t_r]/(t_2^{d_2}, \dots, t_{r-1}^{d_{r-1}}, t_r^{d_r - a_r}) \otimes_K K[t_1, t_{r+1}, \dots, t_s]/(t_{r+1}^{d_{r+1}}, \dots, t_s^{d_s}, t_{r+1}^{a_{r+1}} \cdots t_s^{a_s}).$$

Hence, taking Hilbert series in Eq. (3.1), and applying Lemma 2.7, Theorem 2.5, and Proposition 3.2, we get that the Hilbert series of $S/(L, t^a)$ can be written as

$$F(S/(L, t^a), x) = \frac{x^{a_r}(1 - x^{d_2}) \cdots (1 - x^{d_{r-1}})(1 - x^{d_r - a_r})}{(1 - x)^{r-1}} \frac{g(x)}{(1 - x)} + \frac{(1 - x^{d_2}) \cdots (1 - x^{d_{r-1}})(1 - x^{a_r})(1 - x^{d_{r+1}}) \cdots (1 - x^{d_s})}{(1 - x)^s},$$

where $g(x)/(1 - x)$ is the Hilbert series of the second ring in the tensor product of Eq. (3.2) and $g(1)$ is its degree (see Remark 2.6). By induction hypothesis

$$g(1) = d_{r+1} \cdots d_s - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s).$$

Therefore, writing $F(S/(L, t^a), x) = h(x)/(1 - x)$ with $h(x) \in \mathbb{Z}[x]$ and $h(1) > 0$, and recalling that $h(1)$ is the degree of $S/(L, t^a)$, we get

$$\begin{aligned} \deg(S/(L, t^a)) = h(1) &= d_2 \cdots d_{r-1}(d_r - a_r)g(1) + d_2 \cdots d_{r-1}a_r d_{r+1} \cdots d_s \\ &= d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s). \quad \square \end{aligned}$$

Lemma 3.4. (A) *Let L be the ideal of S generated by $t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p} t_{p+1}^{c_{p+1}}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}$, where $p \geq 1$, $1 \leq c_p \leq c_{p+1}$ and $d_i \geq 1$ for all i . If $t^a = t_1^{a_1} \cdots t_s^{a_s}$ is not in L , $d_{p+1} = c_p + c_{p+1}$, and $a_i \geq 1$ for some i , then the degree of $S/(L, t^a)$ is equal to*

$$\begin{aligned} (i) \quad & d_2 \cdots d_s - (c_{p+1} - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \text{ if } a_p \geq c_p; \\ (ii.1) \quad & d_2 \cdots d_s - (c_p - a_p) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \text{ if } a_p < c_p, a_{p+1} \geq c_{p+1}; \\ (ii.2) \quad & d_2 \cdots d_s - (d_{p+1} - a_p - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \text{ if } a_p < c_p, a_{p+1} < c_{p+1}. \end{aligned}$$

(B) *Let I be a graded ideal such that $L = \text{in}_{\prec}(I)$. If $0 \leq k \leq s - 2$, ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$, then $\text{fp}_I(d) \leq (d_{k+2} - \ell)d_{k+3} \cdots d_s$.*

Proof. (A) Case (i): Assume $a_p \geq c_p$. If $a_i = 0$ for $i \neq p$, then $t^a = t_p^{a_p}$, and by the first equality of Lemma 3.3, and using Lemma 2.7, we get

$$\begin{aligned} \deg(S/(L, t^a)) &= \deg(S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{a_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}, t_p^{c_p} t_{p+1}^{c_{p+1}})) \\ &= \deg(S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s})) = d_2 \cdots d_p c_p d_{p+2} \cdots d_s \\ &= d_2 \cdots d_p (d_{p+1} - c_{p+1}) d_{p+2} \cdots d_s = d_2 \cdots d_s - c_{p+1} d_2 \cdots d_p d_{p+2} \cdots d_s, \end{aligned}$$

as required. We may now assume that $a_i \geq 1$ for some $i \neq p$. As $t^a \notin L$ and $a_p \geq c_p$, one has $a_i < d_{i+1}$ for $i = 1, \dots, p-1$, $a_{p+1} < c_{p+1}$, and $a_i < d_i$ for $i = p+2, \dots, s$. Therefore from the exact sequence

$$\begin{aligned} 0 \longrightarrow S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_{p+1}^{c_{p+1}}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}, t_1^{a_1} \cdots t_{p-1}^{a_{p-1}} t_p^{a_p - c_p} t_{p+1}^{a_{p+1}} \cdots t_s^{a_s})[-c_p] \xrightarrow{t_p^{c_p}} \\ S/(L, t^a) \longrightarrow S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}) \longrightarrow 0, \end{aligned}$$

and using Lemmas 3.3 and 2.7, the required equality follows.

Case (ii): Assume $a_p < c_p$. If $a_i = 0$ for $i \neq p$, then $t^a = t_p^{a_p}$ and $0 = a_{p+1} < c_{p+1}$. Hence, by Lemma 2.7, we get

$$\begin{aligned} \deg(S/(L, t^a)) &= \deg(S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{a_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s})) \\ &= d_2 \cdots d_p a_p d_{p+2} \cdots d_s \\ &= d_2 \cdots d_s - (d_{p+1} - a_p) d_2 \cdots d_p d_{p+2} \cdots d_s, \end{aligned}$$

as required. We may now assume that $a_i \geq 1$ for some $i \neq p$. Consider the exact sequence

$$\begin{aligned} (3.3) \quad 0 \longrightarrow S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_{p+1}^{c_{p+1}}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}, t_1^{a_1} \cdots t_{p-1}^{a_{p-1}} t_{p+1}^{a_{p+1}} \cdots t_s^{a_s})[-c_p] \xrightarrow{t_p^{c_p}} \\ S/(L, t^a) \longrightarrow S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}, t_1^{a_1} \cdots t_s^{a_s}) \longrightarrow 0. \end{aligned}$$

Subcase (ii.1): Assume $a_{p+1} \geq c_{p+1}$. As $t^a \notin L$, in our situation, one has $a_i < d_{i+1}$ for $i = 1, \dots, p-1$, $a_p < c_p$, and $a_i < d_i$ for $i = p+2, \dots, s$. If $a_i = 0$ for $i \neq p+1$, then taking Hilbert series in Eq. (3.3), and noticing that the ring on the right has dimension 0, we get

$$\begin{aligned} \deg(S/(L, t^a)) &= d_2 \cdots d_p c_{p+1} d_{p+2} \cdots d_s \\ &= d_2 \cdots d_s - c_p d_2 \cdots d_p d_{p+2} \cdots d_s, \end{aligned}$$

as required. Thus we may now assume that $a_i \geq 1$ for some $i \neq p+1$. Taking Hilbert series in Eq. (3.3), and using Lemma 2.7, we obtain

$$\begin{aligned} \deg(S/(L, t^a)) &= d_2 \cdots d_p c_{p+1} d_{p+2} \cdots d_s + \\ &\quad \deg(S/(t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s}, t_1^{a_1} \cdots t_s^{a_s})). \end{aligned}$$

Therefore, using Lemma 3.3, the required equality follows.

Subcase (ii.2): Assume $a_{p+1} < c_{p+1}$. If $a_i = 0$ for $i \neq p+1$, taking Hilbert series in Eq. (3.3), and noticing that the ring on the right has dimension 0, by Lemma 3.3, we get

$$\begin{aligned} \deg(S/(L, t^a)) &= d_2 \cdots d_p c_{p+1} d_{p+2} \cdots d_s - d_2 \cdots d_p (c_{p+1} - a_{p+1}) d_{p+2} \cdots d_s \\ &= d_2 \cdots d_p a_{p+1} d_{p+2} \cdots d_s \\ &= d_2 \cdots d_s - (d_{p+1} - a_{p+1}) d_2 \cdots d_p d_{p+2} \cdots d_s, \end{aligned}$$

as required. Thus we may now assume that $a_i \geq 1$ for some $i \neq p+1$. Taking Hilbert series in Eq. (3.3), and applying Lemma 3.3 to the ends of Eq. (3.3), the required equality follows.

(B) It suffices to find a monomial t^b in $\mathcal{M}_{\prec,d}$ such that

$$(3.4) \quad \deg(S/(\text{in}_{\prec}(I), t^b)) = (d_{k+2} - \ell)d_{k+3} \cdots d_s.$$

There are five cases to consider:

$$t^b = \begin{cases} t_1^{d_2-1} \cdots t_{p-1}^{d_p-1} t_p^{c_p} t_{p+1}^{c_{p+1}-1} t_{p+2}^{d_{p+2}-1} \cdots t_{k+1}^{d_{k+1}-1} t_{k+2}^{\ell} & \text{if } k \geq p+1, \\ t_1^{d_2-1} \cdots t_{p-1}^{d_p-1} t_p^{c_p} t_{p+1}^{c_{p+1}-1} t_{p+2}^{\ell} & \text{if } k = p, \\ t_1^{d_2-1} \cdots t_k^{d_{k+1}-1} t_{k+1}^{\ell} & \text{if } k \leq p-2, \\ t_1^{d_2-1} \cdots t_{p-1}^{d_p-1} t_p^{c_p} t_{p+1}^{\ell-c_p} & \text{if } k = p-1 \text{ and } \ell \geq c_p, \\ t_1^{d_2-1} \cdots t_{p-1}^{d_p-1} t_p^{\ell} & \text{if } k = p-1 \text{ and } \ell < c_p. \end{cases}$$

In each case, by the formulas for the degree of part (A), we get the equality of Eq. (3.4). \square

An ideal $I \subset S$ is called *unmixed* if all its associated primes have the same height and I is called *radical* if I is equal to its radical. The radical of I is denoted by $\text{rad}(I)$.

Lemma 3.5. [24, Lemma 4.1] *Let $I \subset S$ be an unmixed graded ideal and let \prec be a monomial order. If $f \in S$ is homogeneous and $(I : f) \neq I$, then*

$$\deg(S/(I, f)) \leq \deg(S/(\text{in}_{\prec}(I), \text{in}_{\prec}(f))) \leq \deg(S/I),$$

and $\deg(S/(I, f)) < \deg(S/I)$ if I is an unmixed radical ideal and $f \notin I$.

Remark 3.6. Let $I \subset S$ be an unmixed graded ideal of dimension 1. If $f \in S_d$, then $(I : f) = I$ if and only if $\dim(S/(I, f)) = 0$. In this case $\deg(S/(I, f))$ could be greater than $\deg(S/I)$.

Lemma 3.7. [24, Lemma 3.2] *Let \mathbb{X} be a finite subset of \mathbb{P}^{s-1} over a field K and let $I(\mathbb{X}) \subset S$ be its graded vanishing ideal. If $0 \neq f \in S$ is homogeneous, then the number of zeros of f in \mathbb{X} is given by*

$$|V_{\mathbb{X}}(f)| = \begin{cases} \deg S/(I(\mathbb{X}), f) & \text{if } (I(\mathbb{X}) : f) \neq I(\mathbb{X}), \\ 0 & \text{if } (I(\mathbb{X}) : f) = I(\mathbb{X}). \end{cases}$$

Corollary 3.8. *Let $I = I(\mathbb{X})$ be the vanishing ideal of a finite set \mathbb{X} of projective points, let $f \in \mathcal{F}_{\prec,d}$, and $\text{in}_{\prec}(f) = t_1^{a_1} \cdots t_s^{a_s}$. If $\text{in}_{\prec}(I)$ is generated by $t_2^{d_2}, \dots, t_s^{d_s}$, then there is $r \geq 2$ such that $a_r \geq 1$, $a_i \leq d_i - 1$ for $i \geq r$, $a_i = 0$ if $2 \leq i < r$, and*

$$|V_{\mathbb{X}}(f)| \leq d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s).$$

Proof. As f is a zero-divisor of S/I , by Lemma 2.8, $t^a = \text{in}_{\prec}(f)$ is a zero-divisor of $S/\text{in}_{\prec}(I)$. Hence, there is $r \geq 2$ such that $a_r \geq 1$ and $a_i = 0$ if $2 \leq i < r$. Using that t^a is a standard monomial of S/I , we get that $a_i \leq d_i - 1$ for $i \geq r$. Therefore, using Lemma 3.7 together with Lemmas 3.3 and 3.5, we get

$$\begin{aligned} |V_{\mathbb{X}}(f)| &= \deg(S/(I(\mathbb{X}), f)) \leq \deg(S/(\text{in}_{\prec}(I(\mathbb{X})), \text{in}_{\prec}(f))) \\ &= d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s). \quad \square \end{aligned}$$

Lemma 3.9. *Let I be a graded ideal and let \prec be a monomial order. Then the minimum distance function δ_I is independent of \prec .*

Proof. Fix a positive integer d . Let \mathcal{F}_d be the set of all homogeneous zero-divisors of S/I not in I of degree d and let f be an element of \mathcal{F}_d . Pick a Gröbner basis g_1, \dots, g_r of I . Then, by the division algorithm [7, Theorem 3, p. 63], we can write $f = \sum_{i=1}^r a_i g_i + h$, where h is a

homogeneous standard polynomial of S/I of degree d . Since $(I : f) = (I : h)$, we get that h is in $\mathcal{F}_{\prec,d}$. Hence, as $(I, f) = (I, h)$, we get the equalities:

$$\begin{aligned}\delta_I(d) &= \deg(S/I) - \max\{\deg(S/(I, f)) \mid f \in \mathcal{F}_{\prec,d}\} \\ &= \deg(S/I) - \max\{\deg(S/(I, f)) \mid f \in \mathcal{F}_d\},\end{aligned}$$

that is, $\delta_I(d)$ does not depends on $\mathcal{F}_{\prec,d}$. \square

Lemma 3.10. *Let I be an unmixed graded ideal and \prec a monomial order. The following hold.*

- (a) $\delta_I(d) \geq \text{fp}_I(d)$ and $\delta_I(d) \geq 0$ for $d \geq 1$.
- (b) $\text{fp}_I(d) \geq 0$ if $\text{in}_{\prec}(I)$ is unmixed.
- (c) $\delta_I(d) \geq 1$ if I is radical.

Proof. If $\mathcal{F}_{\prec,d} = \emptyset$, then clearly $\delta_I(d) = \deg(S/I) \geq 1$, $\delta_I(d) \geq \text{fp}_I(d)$, and if $\text{in}_{\prec}(I)$ is unmixed, then $\text{fp}_I(d) \geq 0$ (this follows from Lemma 3.5). Thus, (a), (b), and (c) hold. Now assume that $\mathcal{F}_{\prec,d} \neq \emptyset$. Pick a standard polynomial $f \in S_d$ such that $(I : f) \neq I$ and

$$\delta_I(d) = \deg(S/I) - \deg(S/(I, f)).$$

As I is unmixed, by Lemma 3.5, $\deg(S/(I, f)) \leq \deg(S/(\text{in}_{\prec}(I), \text{in}_{\prec}(f)))$. On the other hand, by Lemma 2.8, $\text{in}_{\prec}(f)$ is a zero-divisor of $S/\text{in}_{\prec}(I)$. Hence $\delta_I(d) \geq \text{fp}_I(d)$. Using the second inequality of Lemma 3.5 it follows that $\delta_I(d) \geq 0$, $\text{fp}_I(d) \geq 0$ if $\text{in}_{\prec}(I)$ is unmixed, and $\delta_I(d) \geq 1$ if I is radical. \square

Proposition 3.11. *If I is an unmixed monomial ideal and \prec is any monomial order, then $\delta_I(d) = \text{fp}_I(d)$ for $d \geq 1$, that is, I is a Geil–Carvalho ideal.*

Proof. The inequality $\delta_I(d) \geq \text{fp}_I(d)$ follows from Lemma 3.10. To show the reverse inequality notice that $\mathcal{M}_{\prec,d} \subset \mathcal{F}_{\prec,d}$ because one has $I = \text{in}_{\prec}(I)$. Also notice that $\mathcal{M}_{\prec,d} = \emptyset$ if and only if $\mathcal{F}_{\prec,d} = \emptyset$, this follows from Lemma 2.8. Therefore one has $\text{fp}_I(d) \geq \delta_I(d)$. \square

Proposition 3.12. *Let $I \subset S$ be a graded ideal and let \prec be a monomial order. Suppose that $\text{in}_{\prec}(I)$ is a complete intersection of height $s - 1$ generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $d_i \geq 1$ for all i . The following hold.*

- (a) [25, Example 1.5.1] I is a complete intersection and $\dim(S/I) = 1$.
- (b) ([25, Example 1.5.1], [5, Lemma 3.5]) $\deg(S/I) = d_2 \cdots d_s$ and $\text{reg}(S/I) = \sum_{i=2}^s (d_i - 1)$.
- (c) $1 \leq \text{fp}_I(d) \leq \delta_I(d)$ for $d \geq 1$.

Proof. (a): The rings S/I and $S/\text{in}_{\prec}(I)$ have the same dimension. Thus $\dim(S/I) = 1$. As \prec is a graded order, there are f_2, \dots, f_s homogeneous polynomials in I with $\text{in}_{\prec}(f_i) = t^{\alpha_i}$ for $i \geq 2$. Since $\text{in}_{\prec}(I) = (\text{in}_{\prec}(f_2), \dots, \text{in}_{\prec}(f_s))$, the polynomials f_2, \dots, f_s form a Gröbner basis of I , and in particular they generate I . Hence I is a graded ideal of height $s - 1$ generated by $s - 1$ polynomials, that is, I is a complete intersection.

(b): Since I is a complete intersection generated by the f_i 's, then the degree and regularity of S/I are $\deg(f_2) \cdots \deg(f_s)$ and $\sum_{i=2}^s (\deg(f_i) - 1)$, respectively. This follows from the formula for the Hilbert series of a complete intersection given in Lemma 2.7.

(c) The ideal I is unmixed because, by part (a), I is a complete intersection; in particular Cohen–Macaulay and unmixed. Hence the inequality $\delta_I(d) \geq \text{fp}_I(d)$ follows from Lemma 3.10. Let t^a be a standard monomial of S/I of degree d such that $(\text{in}_{\prec}(I) : t^a) \neq \text{in}_{\prec}(I)$, that is, t^a is in $\mathcal{M}_{\prec,d}$. Using Lemma 3.1, and the formulas for $\deg(S/(\text{in}_{\prec}(I), t^a))$ given in Lemma 3.3 and Lemma 3.4, we obtain that $\deg(S/(\text{in}_{\prec}(I), t^a)) < \deg(S/I)$. Thus $\text{fp}_I(d) \geq 1$. \square

Proposition 3.13. [24, Proposition 5.7] *Let $1 \leq e_1 \leq \dots \leq e_m$ and $0 \leq b_i \leq e_i - 1$ for $i = 1, \dots, m$ be integers. If $b_0 \geq 1$, then*

$$(3.5) \quad \prod_{i=1}^m (e_i - b_i) \geq \left(\sum_{i=1}^{k+1} (e_i - b_i) - (k-1) - b_0 - \sum_{i=k+2}^m b_i \right) e_{k+2} \cdots e_m$$

for $k = 0, \dots, m-1$, where $e_{k+2} \cdots e_m = 1$ and $\sum_{i=k+2}^m b_i = 0$ if $k = m-1$.

We come to the main result of this paper.

Theorem 3.14. *Let $I \subset S$ be a graded ideal and let \prec be a graded monomial order. If the initial ideal $\text{in}_{\prec}(I)$ is a complete intersection of height $s-1$ generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then $\delta_I(d) \geq \text{fp}_I(d) \geq 1$ for $d \geq 1$ and*

$$\text{fp}_I(d) = \begin{cases} (d_{k+2} - \ell) d_{k+3} \cdots d_s & \text{if } d \leq \sum_{i=2}^s (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum_{i=2}^s (d_i - 1), \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. Let t^a be any standard monomial of S/I of degree d which is a zero-divisor of $S/\text{in}_{\prec}(I)$, that is, t^a is in $\mathcal{M}_{\prec, d}$. Thus $d = \sum_{i=1}^s a_i$, where $a = (a_1, \dots, a_s)$. We set $r = \sum_{i=2}^s (d_i - 1)$. If we substitute $-\ell = \sum_{i=2}^{k+1} (d_i - 1) - \sum_{i=1}^s a_i$ in the expression $(d_{k+2} - \ell) d_{k+3} \cdots d_s$, it follows that for $d < r$ the inequality

$$\text{fp}_I(d) \geq (d_{k+2} - \ell) d_{k+3} \cdots d_s$$

is equivalent to show that

$$(3.6) \quad \deg(S/I) - \deg(S/(\text{in}_{\prec}(I), t^a)) \geq \left(\sum_{i=2}^{k+2} (d_i - a_i) - k - a_1 - \sum_{i=k+3}^s a_i \right) d_{k+3} \cdots d_s$$

for any t^a in $\mathcal{M}_{\prec, d}$, where by convention $\sum_{i=k+3}^s a_i = 0$ and $d_{k+3} \cdots d_s = 1$ if $k = s-2$. Recall that, by Proposition 3.12, one has that $\text{fp}_I(d) \geq 1$ for $d \geq 1$. By Lemma 3.1, and by permuting variables and changing I , \prec , and t^a accordingly, one has the following two cases to consider.

Case (i): Assume that $\text{in}_{\prec}(I) = (t_2^{d_2}, \dots, t_s^{d_s})$ with $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$. Then, as t^a is in $\mathcal{M}_{\prec, d}$, we can write $t^a = t_1^{a_1} \cdots t_r^{a_r} \cdots t_s^{a_s}$, $r \geq 2$, $a_r \geq 1$, $a_i = 0$ if $2 \leq i < r$, and $a_i \leq d_i - 1$ for $i \geq r$. By Lemma 3.3 we get

$$(3.7) \quad \deg(S/(\text{in}_{\prec}(I), t^a)) = d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s)$$

for any t^a in $\mathcal{M}_{\prec, d}$. If $d \geq r$, setting $t^c = t_1^{d-r} t_2^{d_2-1} \cdots t_s^{d_s-1}$, one has $t^c \in \mathcal{M}_{\prec, d}$. Then, using Eq. (3.7), it follows that $\deg(S/(\text{in}_{\prec}(I), t^c)) = d_2 \cdots d_s - 1$. Thus $\text{fp}_I(d) \leq 1$ and equality $\text{fp}_I(d) = 1$ holds. We may now assume $d \leq r-1$. Setting $t^b = t_2^{d_2-1} \cdots t_{k+1}^{d_{k+1}-1} t_{k+2}^{\ell}$, one has $t^b \in \mathcal{M}_{\prec, d}$. Then, using Eq. (3.7), we get

$$\deg(S/(\text{in}_{\prec}(I), t^b)) = d_2 \cdots d_s - (d_{k+2} - \ell) d_{k+3} \cdots d_s.$$

Hence $\text{fp}_I(d) \leq (d_{k+2} - \ell) d_{k+3} \cdots d_s$. Next we show the reverse inequality by showing that the inequality of Eq. (3.6) holds for any $t^a \in \mathcal{M}_{\prec, d}$. By Eq. (3.7) it suffices to show that the following equivalent inequality holds

$$(d_2 - a_2) \cdots (d_s - a_s) \geq \left(\sum_{i=2}^{k+2} (d_i - a_i) - k - a_1 - \sum_{i=k+3}^s a_i \right) d_{k+3} \cdots d_s$$

for any $a = (a_1, \dots, a_s)$ such that $t^a \in \mathcal{M}_{\prec, d}$. This inequality follows from Proposition 3.13 by making $m = s - 1$, $e_i = d_{i+1}$, $b_i = a_{i+1}$ for $i = 1, \dots, s - 1$ and $b_0 = 1 + a_1$.

Case (ii): Assume that $\text{in}_{\prec}(I) = (t_1^{d_2}, \dots, t_{p-1}^{d_p}, t_p^{c_p} t_{p+1}^{c_{p+1}}, t_{p+2}^{d_{p+2}}, \dots, t_s^{d_s})$ for some $p \geq 1$ such that $1 \leq c_p \leq c_{p+1}$ and $1 \leq d_i \leq d_{i+1}$ for all i , where $d_{p+1} = c_p + c_{p+1}$.

If $d \geq r$, setting $t^c = t_1^{d_2-1} \dots t_{p-1}^{d_p-1} t_p^{d-r+c_p} t_{p+1}^{c_{p+1}-1} t_{p+2}^{d_{p+2}-1} \dots t_s^{d_s-1}$, we get that $t^c \in \mathcal{M}_{\prec, d}$. Then, using the first formula of Lemma 3.4, it follows that $\deg(S/(\text{in}_{\prec}(I), t^c)) = d_2 \cdots d_s - 1$. Thus $\text{fp}_I(d) \leq 1$ and the equality $\text{fp}_I(d) = 1$ holds.

We may now assume $d \leq r - 1$. The inequality $\text{fp}_I(d) \leq (d_{k+2} - \ell) d_{k+3} \cdots d_s$ follows from Lemma 3.4(B). To show that $\text{fp}_I(d) \geq (d_{k+2} - \ell) d_{k+3} \cdots d_s$ we need only show that the inequality of Eq. (3.6) holds for any t^a in $\mathcal{M}_{\prec, d}$. Take t^a in $\mathcal{M}_{\prec, d}$. Then we can write $t^a = t_1^{a_1} \cdots t_s^{a_s}$ with $a_i < d_{i+1}$ for $i < p$ and $a_i < d_i$ for $i > p + 1$. There are three subcases to consider.

Subcase (ii.1): Assume $a_p \geq c_p$. Then $c_{p+1} > a_{p+1}$ because t^a is a standard monomial of S/I , and by Lemma 3.4 we get

$$\deg(S/(\text{in}_{\prec}(I), t^a)) = d_2 \cdots d_s - (c_{p+1} - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i).$$

Therefore the inequality of Eq. (3.6) is equivalent to

$$\begin{aligned} & (c_{p+1} - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \\ & \geq \left(\sum_{i=2}^{k+2} (d_i - a_i) - k - a_1 - \sum_{i=k+3}^s a_i \right) d_{k+3} \cdots d_s, \end{aligned}$$

and this inequality follows at once from Proposition 3.13 by making $m = s - 1$, $e_i = d_{i+1}$ for $i = 1, \dots, m$, $b_i = a_i$ for $1 \leq i \leq p - 1$, $b_p = a_{p+1} + c_p$, $b_i = a_{i+1}$ for $p < i \leq m$, and $b_0 = a_p - c_p + 1$. Notice that $\sum_{i=0}^m b_i = 1 + \sum_{i=1}^s a_i$.

Subcase (ii.2): Assume $a_p < c_p$, $a_{p+1} \geq c_{p+1}$. By Lemma 3.4 we get

$$\deg(S/(\text{in}_{\prec}(I), t^a)) = d_2 \cdots d_s - (c_p - a_p) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i).$$

Therefore the inequality of Eq. (3.6) is equivalent to

$$\begin{aligned} & (c_p - a_p) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \\ & \geq \left(\sum_{i=2}^{k+2} (d_i - a_i) - k - a_1 - \sum_{i=k+3}^s a_i \right) d_{k+3} \cdots d_s, \end{aligned}$$

and this inequality follows from Proposition 3.13 by making $m = s - 1$, $e_i = d_{i+1}$ for $i = 1, \dots, m$, $b_i = a_i$ for $1 \leq i \leq p - 1$, $b_p = c_{p+1} + a_p$, $b_i = a_{i+1}$ for $p < i \leq m$, and $b_0 = a_{p+1} - c_{p+1} + 1$. Notice that $\sum_{i=0}^m b_i = 1 + \sum_{i=1}^s a_i$.

Subcase (ii.3): Assume $a_p < c_p$, $a_{p+1} \leq c_{p+1} - 1$. By Lemma 3.4 we get

$$\deg(S/(\text{in}_{\prec}(I), t^a)) = d_2 \cdots d_s - (d_{p+1} - a_p - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i).$$

Therefore the inequality of Eq. (3.6) is equivalent to

$$\begin{aligned} & (d_{p+1} - a_p - a_{p+1}) \prod_{i=1}^{p-1} (d_{i+1} - a_i) \prod_{i=p+2}^s (d_i - a_i) \\ & \geq \left(\sum_{i=2}^{k+2} (d_i - a_i) - k - a_1 - \sum_{i=k+3}^s a_i \right) d_{k+3} \cdots d_s, \end{aligned}$$

and this inequality follows from Proposition 3.13 by making $m = s-1$, $e_i = d_{i+1}$ for $i = 1, \dots, m$, $b_i = a_i$ for $1 \leq i \leq p-1$, $b_p = a_p + a_{p+1}$, $b_i = a_{i+1}$ for $p < i \leq m$, and $b_0 = 1$. Notice that in this case $\sum_{i=0}^m b_i = 1 + \sum_{i=1}^s a_i$. \square

4. APPLICATIONS AND EXAMPLES

This section is devoted to give some applications and examples of our main result. As the two most important applications to algebraic coding theory, we recover the formula for the minimum distance of an affine cartesian code [13, 21], and the fact that the homogenization of the corresponding vanishing ideal is a Geil–Carvalho ideal [4].

We begin with a basic application for complete intersections in \mathbb{P}^1 .

Corollary 4.1. *If \mathbb{X} is a finite subset of \mathbb{P}^1 and $I(\mathbb{X})$ is a complete intersection, then*

$$\delta_{I(\mathbb{X})}(d) = \text{fp}_{I(\mathbb{X})}(d) = \begin{cases} |\mathbb{X}| - d & \text{if } 1 \leq d \leq |\mathbb{X}| - 2, \\ 1 & \text{if } d \geq |\mathbb{X}| - 1. \end{cases}$$

Proof. Let f be the generator of $I(\mathbb{X})$. In this case $d_2 = \deg(f) = |\mathbb{X}|$ and $\text{reg}(S/I(\mathbb{X})) = |\mathbb{X}| - 1$. By Proposition 2.12 and Theorem 3.14 one has

$$\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \geq \text{fp}_{I(\mathbb{X})}(d) = |\mathbb{X}| - d \quad \text{for } 1 \leq d \leq |\mathbb{X}| - 2,$$

and $\delta_{\mathbb{X}}(d) = 1$ for $d \geq |\mathbb{X}| - 1$. Assume that $1 \leq d \leq |\mathbb{X}| - 2$. Pick $[P_1], \dots, [P_d]$ points in \mathbb{P}^1 . By Lemma 2.14, the vanishing ideal $I_{[P_i]}$ of $[P_i]$ is a principal ideal generated by a linear form h_i . Notice that $V_{\mathbb{X}}(h_i)$, the zero-set of h_i in \mathbb{X} , is equal to $\{[P_i]\}$. Setting $h = h_1 \cdots h_d$, we get a homogeneous polynomial of degree d with exactly d zeros. Thus $\delta_{\mathbb{X}}(d) \leq |\mathbb{X}| - d$. \square

As another application we get the following uniform upper bound for the number of zeros all polynomials $f \in S_d$ that do not vanish at all points of \mathbb{X} .

Corollary 4.2. *Let \mathbb{X} be a finite subset of \mathbb{P}^{s-1} , let $I(\mathbb{X})$ be its vanishing ideal, and let \prec be a monomial order. If the initial ideal $\text{in}_{\prec}(I(\mathbb{X}))$ is a complete intersection generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then*

$$(4.1) \quad |V_{\mathbb{X}}(f)| \leq \deg(S/I(\mathbb{X})) - (d_{k+2} - \ell) d_{k+3} \cdots d_s,$$

for any $f \in S_d$ that does not vanish at all point of \mathbb{X} , where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. It follows from Corollary 2.13, Eq. (2.2), and Theorem 3.14. \square

We leave as an open question whether this uniform bound is optimal, that is, whether the equality is attained for some polynomial f . Another open question is whether Corollary 4.2 is true if we only assume that $I(\mathbb{X})$ is a complete intersection. This is related to the following conjecture of Tohăneanu and Van Tuyl.

Conjecture 4.3. [33, Conjecture 4.9] *Let \mathbb{X} be a finite set of points in \mathbb{P}^{s-1} . If $I(\mathbb{X})$ is a complete intersection generated by f_1, \dots, f_{s-1} , with $e_i = \deg(f_i)$ for $i = 1, \dots, s-1$, and $2 \leq e_i \leq e_{i+1}$ for all i , then $\delta_{\mathbb{X}}(1) \geq (e_1 - 1)e_2 \cdots e_{s-1}$.*

Notice that by Corollary 4.2 this conjecture is true if $\text{in}_{\prec}(I(\mathbb{X}))$ is a complete intersection, and it is also true for $s = 2$ (see Corollary 4.1).

Affine cartesian codes and coverings by hyperplanes. Given a collection of finite subsets A_2, \dots, A_s of a field K , we denote the image of

$$X^* = A_2 \times \cdots \times A_s$$

under the map $\mathbb{A}^{s-1} \mapsto \mathbb{P}^{s-1}$, $x \mapsto [(1, x)]$, by $\mathbb{X} = [1 \times A_2 \times \cdots \times A_s]$. The affine Reed-Muller-type code $C_{\mathbb{X}^*}(d)$ of degree d is called an *affine cartesian code* [21]. The basic parameters of the projective Reed-Muller-type code $C_{\mathbb{X}}(d)$ are equal to those of $C_{\mathbb{X}^*}(d)$ [22].

A formula for the minimum distance of an affine cartesian code is given in [21, Theorem 3.8] and in [13, Proposition 5]. A short and elegant proof of this formula was given by Carvalho in [4, Proposition 2.3], where he shows that the best way to study the minimum distance of an affine cartesian code is by using the footprint. As an application of Theorem 3.14 we also recover the formula for the minimum distance of an affine cartesian code by examining the underlying vanishing ideal and show that this ideal is Geil–Carvalho.

Corollary 4.4. [4, 13, 21] *Let K be a field and let $C_{\mathbb{X}}(d)$ be the projective Reed-Muller type code of degree d on the finite set $\mathbb{X} = [1 \times A_2 \times \cdots \times A_s] \subset \mathbb{P}^{s-1}$. If $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, with $d_i = |A_i|$, and $d \geq 1$, then the minimum distance of $C_{\mathbb{X}}(d)$ is given by*

$$\delta_{\mathbb{X}}(d) = \begin{cases} (d_{k+2} - \ell) d_{k+3} \cdots d_s & \text{if } d \leq \sum_{i=2}^s (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum_{i=2}^s (d_i - 1), \end{cases}$$

and $I(\mathbb{X})$ is Geil–Carvalho, that is, $\delta_{I(\mathbb{X})}(d) = \text{fp}_{I(\mathbb{X})}(d)$ for $d \geq 1$, where $k \geq 0$, ℓ are the unique integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. Let \succ be the reverse lexicographical order on S with $t_2 \succ \cdots \succ t_s \succ t_1$. Setting $f_i = \prod_{\gamma \in A_i} (t_i - \gamma t_1)$ for $i = 2, \dots, s$, one has that f_2, \dots, f_s is a Gröbner basis of $I(\mathbb{X})$ whose initial ideal is generated by $t_2^{d_2}, \dots, t_s^{d_s}$ (see [21, Proposition 2.5]). By Theorem 2.13 one has the equality $\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d)$ for $d \geq 1$. Thus the inequality “ \geq ” follows at once from Theorem 3.14. This is the difficult part of the proof. The rest of the argument reduces to finding an appropriate polynomial f where equality occurs, and to using that the minimum distance $\delta_{\mathbb{X}}(d)$ is 1 for d greater than or equal to $\text{reg}(S/I(\mathbb{X}))$.

We set $r = \sum_{i=2}^s (d_i - 1)$. By Propositions 2.12 and 3.12, the regularity and the degree of $S/I(\mathbb{X})$ are r and $|\mathbb{X}| = d_2 \cdots d_s$, respectively. Assume that $d < r$. To show the inequality “ \leq ” notice that there is a polynomial $f \in S_d$ which is a product of linear forms such that $|V_{\mathbb{X}}(f)|$, the number of zeros of f in \mathbb{X} , is equal to

$$d_2 \cdots d_s - (d_{k+2} - \ell) d_{k+3} \cdots d_s,$$

see [21, p. 15]. Hence $\delta_{\mathbb{X}}(d)$ is less than or equal to $(d_{k+2} - \ell) d_{k+3} \cdots d_s$. Thus the required equality holds. If $d \geq r$, by Proposition 2.12, $\delta_{\mathbb{X}}(d) = 1$ for $d \geq r$. Therefore, by Theorem 3.14, $I(\mathbb{X})$ is Geil–Carvalho. \square

The next result is an extension of a result of Alon and Füredi [1, Theorem 1] that can be applied to any finite subset of a projective space whose vanishing ideal has a complete intersection initial ideal relative to a graded monomial order.

Corollary 4.5. *Let \mathbb{X} be a finite subset of a projective space \mathbb{P}^{s-1} and let \prec be a monomial order such that $\text{in}_\prec(I(\mathbb{X}))$ is a complete intersection generated by $t^{\alpha_2}, \dots, t^{\alpha_s}$, with $d_i = \deg(t^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for all i . If the hyperplanes H_1, \dots, H_d in \mathbb{P}^{s-1} avoid a point $[P]$ in \mathbb{X} but otherwise cover all the other $|\mathbb{X}| - 1$ points of \mathbb{X} , then $d \geq \text{reg}(S/I(\mathbb{X})) = \sum_{i=2}^s (d_i - 1)$.*

Proof. Let h_1, \dots, h_d be the linear forms in S_1 that define H_1, \dots, H_d , respectively. Assume that $d < \sum_{i=2}^s (d_i - 1)$. Consider the polynomial $h = h_1 \cdots h_d$. Notice that $h \notin I(\mathbb{X})$ because $h(P) \neq 0$, and $h(Q) = 0$ for all $[Q] \in \mathbb{X}$ with $[Q] \neq [P]$. By Theorem 3.14, $\delta_{\mathbb{X}}(d) \geq \text{fp}_{I(\mathbb{X})}(d) \geq 2$. Hence, h does not vanish in at least two points of \mathbb{X} , a contradiction. \square

Example 4.6. Let S be the polynomial ring $\mathbb{F}_3[t_1, t_4, t_3, t_2]$ with the lexicographical order $t_1 \prec t_4 \prec t_3 \prec t_2$, and let $I = I(\mathbb{X})$ be the vanishing ideal of

$$\begin{aligned} \mathbb{X} = \{ & [(1, 0, 0, 0)], [(1, 1, 1, 0)], [(1, -1, -1, 0)], [(1, 1, 0, 1)], \\ & [(1, -1, 1, 1)], [(1, 0, -1, 1)], [(1, -1, 0, -1)], [(1, 0, 1, -1)], [(1, 1, -1, -1)] \}. \end{aligned}$$

Using the procedure below in *Macaulay2* [16] and Theorem 3.14, we obtain the following information. The ideal $I(\mathbb{X})$ is generated by $t_2 - t_3 - t_4$, $t_3^3 - t_3 t_1^2$, and $t_4^3 - t_4 t_1^2$. The regularity and the degree of $S/I(\mathbb{X})$ are 4 and 9, respectively, and $I(\mathbb{X})$ is a Geil–Carvalho ideal whose initial ideal is a complete intersection generated by t_2 , t_3^3 , t_4^3 . The basic parameters of the Reed–Muller-type code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3	4
$ \mathbb{X} $	9	9	9	9
$H_{\mathbb{X}}(d)$	3	6	8	9
$\delta_{\mathbb{X}}(d)$	6	3	2	1
$\text{fp}_{I(\mathbb{X})}(d)$	6	3	2	1

By Corollary 4.5, if the hyperplanes H_1, \dots, H_d in \mathbb{P}^3 avoid a point $[P]$ in \mathbb{X} but otherwise cover all the other $|\mathbb{X}| - 1$ points of \mathbb{X} , then $d \geq \text{reg}(S/I(\mathbb{X})) = 4$.

```
S=ZZ/3[t2,t3,t4,t1,MonomialOrder=>Lex];
I1=ideal(t2,t3,t4),I2=ideal(t4,t3-t1,t2-t1),I3=ideal(t4,t1+t3,t2+t1)
I4=ideal(t4-t1,t4-t2,t3),I5=ideal(t4-t1,t3-t1,t2+t1),I6=ideal(t2,t1-t4,t3+t1)
I7=ideal(t3,t1+t4,t1+t2),I8=ideal(t2,t4+t1,t3-t1),I9=ideal(t1+t4,t3+t1,t2-t1)
I=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9)
M=coker gens gb I, regularity M, degree M
h=(d)->degree M - max apply(apply(apply(apply (toList
(set(0..q-1))** (hilbertFunction(d,M))-(set{0})** (hilbertFunction(d,M))),
toList),x->basis(d,M)*vector x),z->ideal(flatten entries z)),
x-> if not quotient(I,x)==I then degree ideal(I,x) else 0)--this
--gives the minimum distance in degree d
apply(1..3,h)
```

Example 4.7. Let S be the polynomial ring $S = \mathbb{F}_3[t_1, t_2, t_3]$ with the lexicographical order $t_1 \succ t_2 \succ t_3$, and let $I = I(\mathbb{X})$ be the vanishing ideal of

$$\mathbb{X} = \{ [(1, 1, 0)], [(1, -1, 0)], [(1, 0, 1)], [(1, 0, -1)], [(1, -1, -1)], [(1, 1, 1)] \}.$$

As in Example 4.6, using *Macaulay2* [16], we get that $I(\mathbb{X})$ is generated by

$$t_2^2 t_3 - t_2 t_3^2, \quad t_1^2 - t_2^2 + t_2 t_3 - t_3^2.$$

The regularity and the degree of $S/I(\mathbb{X})$ are 3 and 6, respectively, I is a Geil–Carvalho ideal, and $\text{in}_<(I)$ is a complete intersection generated by $t_2^2 t_3$ and t_1^2 . The basic parameters of the Reed-Muller-type code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3
$ \mathbb{X} $	6	6	6
$H_{\mathbb{X}}(d)$	3	5	6
$\delta_{\mathbb{X}}(d)$	3	2	1
$\text{fp}_{I(\mathbb{X})}(d)$	3	2	1

By Corollary 4.5, if the hyperplanes H_1, \dots, H_d in \mathbb{P}^2 avoid a point $[P]$ in \mathbb{X} but otherwise cover all the other $|\mathbb{X}| - 1$ points of \mathbb{X} , then $d \geq \text{reg}(S/I(\mathbb{X})) = 3$.

Next we give an example of a graded vanishing ideal over a finite field, which is not Geil–Carvalho, by computing all possible initial ideals.

Example 4.8. Let $\mathbb{X} = \mathbb{P}^2$ be the projective space over the field \mathbb{F}_2 and let $I = I(\mathbb{X})$ be the vanishing ideal of \mathbb{X} . Using the procedure below in *Macaulay2* [16] we get that the binomials $t_1 t_2^2 - t_1^2 t_2$, $t_1 t_3^2 - t_1^2 t_3$, $t_2 t_3^2 - t_2^2 t_3$ form a universal Gröbner basis of I , that is, they form a Gröbner basis for any monomial order. The ideal I has exactly six different initial ideals and $\delta_{\mathbb{X}} \neq \text{fp}_I$ for each of them, that is, I is not a Geil–Carvalho ideal. The basic parameters of the projective Reed-Muller code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3
$ \mathbb{X} $	7	7	7
$H_{\mathbb{X}}(d)$	3	6	7
$\delta_{\mathbb{X}}(d)$	4	2	1
$\text{fp}_{I(\mathbb{X})}(d)$	4	1	1

```
load "gfaninterface.m2"
S=ZZ/2[symbol t1, symbol t2, symbol t3]
I=ideal(t1*t2^2-t1^2*t2,t1*t3^2-t1^2*t3,t2*t3^2-t2^2*t3)
universalGroebnerBasis(I)
(InL,L)= gfan I, #InL
init=ideal(InL_0)
M=coker gens gb init
f=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
fp=(d) ->degree M -max apply(flatten entries basis(d,M),f)
apply(1..regularity(M),fp)
```

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